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Complex Diffusive Plankton System with Monod-Haldane Functional Response: Effect of Prey-Taxis

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--*ABSTRACT*--
General reaction-diffusion equation comprises of the prey-taxis phenomenon. Prey-taxis express the active movement of predator species in the direction of prey due to high concentration of prey or some sort of cue. We propose a reaction-diffusion-advection equation for plankton interaction. Analytically we discuss the local stability of system dynamics. Our numerical investigation reveals that increasing active movement of the predator shows chaotic behaviour of the system. Prey ability of better defense reduces the predator density.
KEYWORDS: *Phytoplankton-Zooplankton, Monod-Haldane functional response, Prey-taxis, Spatio-temporal pattern*

I. INTRODUCTION

Modelling complex plankton system has many aspects of human interest as it is the base food chain of aquatic and wetland ecosystem [1]. Examining the functional properties of different phytoplankton groups can dominates is provided by the model [2]. Wang *et al*. [3] studied a nutrient-algae interaction model for wetland in presence of diffusion. Contamination of water bodies with oocysts of *Cryptosporidium parvum* causes numerous waterborne outbreaks of this parasitic protozoan disease worldwide [4]. Chakraborty *et al*. [5] discussed the effect of prey-taxis on *Paramecium aurelia* and *Didiniumnasutum* interaction through a diffusive predator prey model and observed that the active predator velocity helps to stabilize the system. A prey-taxis model for high prey density areas is discussed by Kareiva & Odell [6]. Further the model system is investigated for mean travel time of a predator to reach a prey resource [7]. It has been observed that impassable prey taxis destroy the stability, leads to chaotic dynamics [8]. Tyutyunov *et al*. [9] has studied a pursuit-evasion model with prey-taxis and observed the continuous travelling waves. Lee *et al*. [10-11] has discussed the Allee effect on prey dynamics. Chakraborty *et al*.[12-13] studied two-spotted spider mite model with effect of prey-taxis and periodic, quasi-periodic and chaotic natures of solution have been observed. It has been observed that the small value of prey-taxis and increasing random movement stabilize system dynamics [14].

We propose a mathematical model for interaction of wetland species phytoplankton and zooplankton with prey taxis and diffusion. Functional response of predator is still the basis of predator- prey theory [15]. A simplified Monod–Haldane function is considered for inverse measure of inhibitory effect. The paper has organized in 6 sections. In Section 2, we have described the model system. Sections 3 and 4 depict the stability analysis in absence and presence of diffusion respectively. Results of numerical simulations are presented and discussed in section 5. Section 6 concludes this study.

II. MODEL SYSTEM

A phytoplankton-zooplankton interaction model where phytoplankton $X(x,t)$, zooplankton $Y(x,t)$ population density and $z(x,t)$ be the velocity of zooplankton at any location x and time t. The directed zooplankton movement is described as [16]:

$$\left(\frac{\partial}{\partial t} + z\cdot\nabla\right)\mathrm{z} = \mu\nabla X,$$

where μ is the positive taxis coefficient.

Introducing diffusion in the zooplankton velocity equation, we get

$$\left(\frac{\partial}{\partial t} + z\cdot\nabla\right)\mathrm{z} = \mu\nabla X + d_3\nabla^2 z,$$

neglecting $z\cdot\nabla$ from above equation as it is sufficiently small, we write

$$\left(\frac{\partial z}{\partial t}\right) = \mu\frac{\partial X}{\partial x} + d_3\frac{\partial^2 z}{\partial x^2}, \tag{1}$$

where μ is the taxis coefficient of the zooplankton.
We consider the reaction-diffusion-advection model system as

$$\frac{\partial X}{\partial t} = rX\left(1-\frac{X}{K}\right) - \frac{aXY}{d+bX^2} + d_1\frac{\partial^2 X}{\partial x^2}, \tag{2a}$$

$$\frac{\partial Y}{\partial t} = \frac{aeXY}{d+bX^2} - cY - \frac{\partial (Yz)}{\partial x} + d_2\frac{\partial^2 Y}{\partial x^2}. \tag{2b}$$

Choosing the variables $\tilde{t} = rt$, $\tilde{X} = \frac{X}{K}, \tilde{Y} = \frac{Y}{eK}$ and $\tilde{z} = \frac{z}{r}$; setting the parameters $w = \frac{aeK}{dr}, w_1 = \frac{bK^2}{d}, c_1 = \frac{c}{r}, \tilde{\mu} = \frac{\mu K}{r^2}, \widetilde{d_1} = \frac{d_1}{r}, \widetilde{d_2} = \frac{d_2}{r}$, and $\widetilde{d_3} = \frac{d_3}{r}$; and dropping ~ for notational convenience. We have the following dimensionless form

$$\frac{\partial X}{\partial t} = X(1-X) - \frac{wXY}{1+w_1X^2} + d_1\frac{\partial^2 X}{\partial x^2}, \tag{3a}$$

$$\frac{\partial Y}{\partial t} = \frac{wXY}{1+w_1X^2} - c_1Y - \frac{\partial (Yz)}{\partial x} + d_2\frac{\partial^2 Y}{\partial x^2}, \tag{3b}$$

$$\frac{\partial z}{\partial t} = \mu\frac{\partial X}{\partial x} + d_3\frac{\partial^2 z}{\partial x^2}. \tag{3c}$$

With zero flux boundary conditions

$$z|_{x=0,L} = \left.\frac{\partial X}{\partial x}\right|_{x=0,L} = \left.\frac{\partial Y}{\partial x}\right|_{x=0,L} = 0. \tag{4}$$

III. STABILITY ANALYSIS OF A TEMPORAL MODEL

Firstly, we analyze (3a)-(3b) in absence of diffusion

$$\frac{dX}{dt} = X(1-X) - \frac{wXY}{1+w_1X^2}, \tag{5a}$$

$$\frac{dY}{dt} = \frac{wXY}{1+w_1X^2} - c_1Y. \tag{5b}$$

The three non-negative equilibria are $E_0(0,0), E_1(1,0)$ and $E^*(X^*,Y^*)$. The $E^*(X^*,Y^*)$ is the point of intersection of two nullclines $wY = (1-X)(1+w_1X^2)$ and $wX = c_1(1+w_1X^2)$, where X^* is the positive root of the equation $X^{*2} + pX^* + q = 0$, with the values of $p = -\frac{w}{w_1c_1}$ and $q = \frac{1}{w}$ and $Y^* = \frac{1}{w}(1-X)(1+$ $w1X2$ respectively.

We discuss the local behavior of the system by computing variational matrices and obtained the followings

(i) E_0 is a saddle point.
(ii) If $w < c_1(1+w_1), E_1$ is locally asymptotically.
(iii) If $w > c_1(1+w_1), E_1$ is a saddle point.

We have

$$A' = \left(1 - \frac{2ww_1X^*Y^*}{\left(1+w_1X^{*2}\right)^2}\right)X^*, \tag{6}$$

$$B' = \frac{w^2X^*Y^*(1-w_1X^2)}{\left(1+w_1X^{*2}\right)^3}. \tag{7}$$

Theorem 3.1. *The unique non-trivial positive equilibrium point* $E^*(X^*,Y^*)$ *is locally asymptotically stable if and only if the following inequalities hold*:

(i) $\left(1+w_1X^{*2}\right)^2 > 2ww_1X^*Y^*$, (8)

(ii) $X^{*2} < 1/w_1$. (9)

The proof is omitted as it follows the Routh-Hurwitz criteria.

Lemma 3.2. *Solution of system*$5(a)$ and $5(b)$*with a positive initial condition remains positive and bounded. Furthermore, there exists* $t' \geq 0$ *such that* $X(t) < 1$ *and* $\left(X(t)+Y(t)\right) \leq \frac{2}{\varphi}$ *for* $t \geq t'$, *where* $\varphi = min(1, c_1)$.

The proof of this lemma can be obtained by simple calculation and hence omitted.

IV. STABILITY ANALYSIS OF THE SPATIAL MODEL SYSTEM

In this section linearized system (3) about $E^*(X^*,Y^*,0)$ with small perturbations *U(x, t), V(x, t),* and *W(x, t)* as follows:

$$\frac{\partial U}{\partial t} = a_{11}U + a_{12}V + d_1\frac{\partial^2 U}{\partial x^2}, \tag{10a}$$

$$\frac{\partial V}{\partial t} = a_{21}U + a_{22}V - Y^*\frac{\partial W}{\partial x} + d_2\frac{\partial^2 V}{\partial x^2}, \tag{10b}$$

$$\frac{\partial W}{\partial t} = \mu\frac{\partial U}{\partial x} + d_3\frac{\partial^2 W}{\partial x^2}, \tag{10c}$$

where

$a_{11} = \left(-1 + \frac{2ww_1X^*Y^*}{\left(1+w_1X^{*2}\right)^2}\right)X^*, a_{12} = \frac{-wX^*}{1+w_1X^{*2}},\ a_{21} = \left(\frac{w\left(1-w_1X^{*2}\right)}{\left(1+w_1X^{*2}\right)^2}\right)Y^*, a_{22} = 0.$

Let us take the solutions of system (10) in the form

$U(x,t) = \sum_K U_k e^{\lambda t} \cos kx,$ (11a)

$V(x,t) = \sum_K V_k e^{\lambda t} \cos kx,$ (11b)

$W(x,t) = \sum_K W_k e^{\lambda t} \sin kx,$ (11c)

where $k = n\pi/L$ is the wave number. The characteristic equation of the linearized system is given by

$\lambda^3 + p(k^2)\lambda^2 + q(k^2)\lambda + r(k^2) = 0,$ (12)

where

$p(k^2) = k^2(d_1 + d_2 + d_3) - a_{11},$ (13a)

$q(k^2) = k^4(d_1d_2 + d_2d_3 + d_3d_1) - k^2(d_2 + d_3)a_{11} - a_{12}a_{21},$ (13b)

$r(k^2) = k^2(d_1d_2d_3k^4 - a_{11}d_2d_3k^2 - a_{12}a_{21}d_3 - a_{12}Y^*\mu),$ (13c)

Applying Routh-Hurwitz criteria, the stability of $E^*(X^*, Y^*, 0)$ depends on

$p(k^2) > 0,$ (14a)

$r(k^2) > 0,$ (14b)

$p(k^2)q(k^2) - r(k^2) > 0.$ (14c)

Using (12)-(13) and Routh-Hurwitz criteria, the theorem follows

Theorem 4.1. *If Eq.* (8) *and* (9) *are satisfied the positive equilibrium* E^* *is locally asymptotically stable in the presence of diffusion and taxis if and only if*

$0 \le \mu < \frac{1+w_1X^{*2}}{wk^2X^*Y^*}\left(s(k^2) + p(k^2)q(k^2)\right),$ (15)

where $s(k^2) = k^2\left(a_{12}a_{21}d_3 + d_2d_3k^2(a_{11} - d_1k^2)\right).$

The Turing instability occurred if at least one of the roots of Eq. (12) have a positive roots or positive real part i.e., $\mathrm{Re}(\lambda) > 0$ for some k≠ 0. From Eq. (13), we observe that the diffusion driven instability occur when $p(k^2)q(k^2) - r(k^2) < 0$. Hence the corresponding condition is given by

$H(k^2) = p(k^2)q(k^2) - r(k^2) < 0.$ (16)

The graph of the function $\mathrm{H}(k^2)$ versus k^2 has been plotted for $\mu = 0.1\ to\ 0.3$ (c.f. Fig. (1)). $\mathrm{H}(k^2) < 0$ shows the Turing instability region.

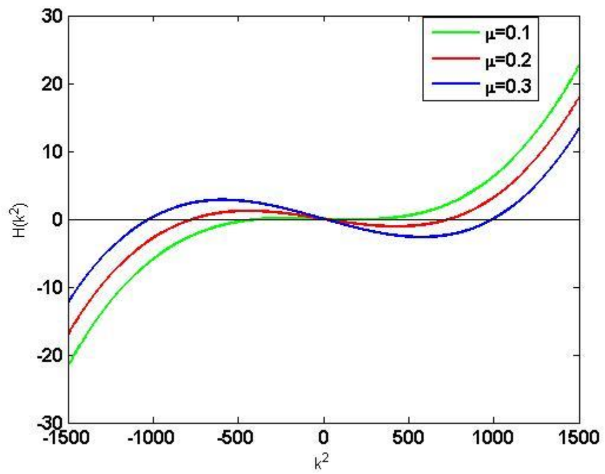
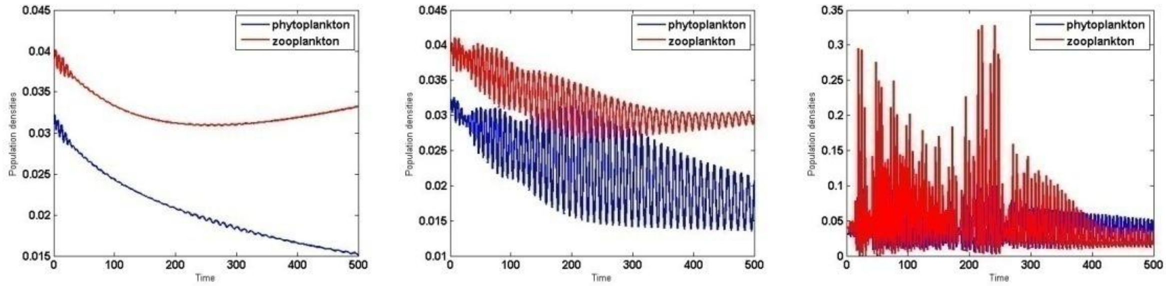


Fig.1. The graph of $\mathrm{H}(k^2)$ versus k^2 for $w = 24.93, w_1 = 5.15, c_1 = 0.8, d_1 = 0.001, d_2 = 0.002, d_3 = 0.0001$ and different value of prey-taxis $\mu = 0.1, 0.2, 0.3$.

V. NUMERICAL SIMULATIONS RESULTS

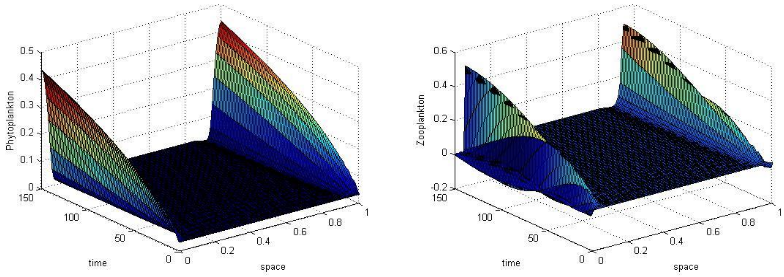
The numerical simulation is performed on MATLAB 7.14 platform. We have taken $w = 24.93, w_1 = 5.15, c_1 = 0.8$, for the model system (5).For the above set of parameters, we obtained $(X^*, Y^*, 0) = (0.0322, 0.0390, 0)$. For the fixed values of parameters $w = 24.93, w_1 = 5.15, c_1 = 0.8, d_1 = 0.001, d_2 = 0.002, d_3 = 0.0001, k = \frac{3\pi}{5}$, the critical value of prey-taxis parameter $\mu = 0.2251$. Our objectives to investigate the effect of prey-taxis (μ) and inhibitory effect (w_1). In Fig. 2, we have plotted the time series for μ = 0.09, 0.12, 0.35. We observe that the increasing value of the prey-taxis system shows stable to limit cycle and finally, it becomes chaotic.



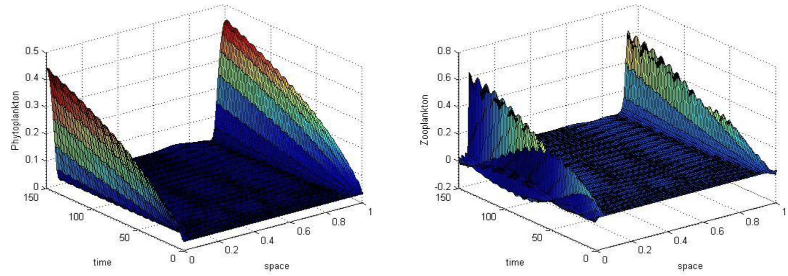
(a) (b) (c)

Fig. 2. Time series of model system (3) with (a) μ =0.09, (b) μ =0.12, (c) μ =0.35.

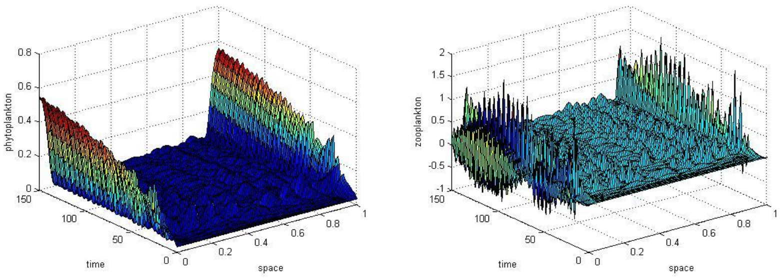
The Spatio-temporal patterns have been plotted to understand the effect of prey-taxis (μ) and inhibitory effect (w_1). In Fig.3, we noticed that the accumulate value of $\mu = 0.09$ to 0.35, the system reveals irregular chaotic dynamics. Similarly, accumulate the value of $w_1 = 5.45$, 5.95, the system again admits irregular chaotic dynamics (c.f. Fig. 4).



(a)

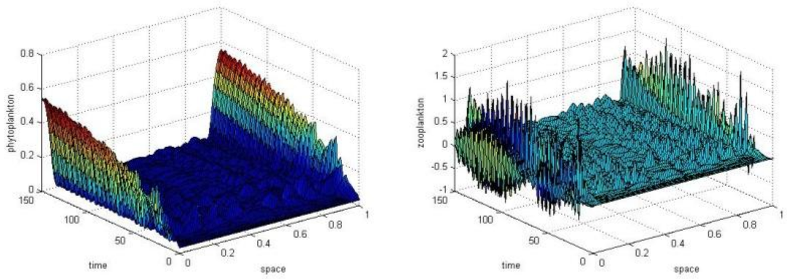


(b)

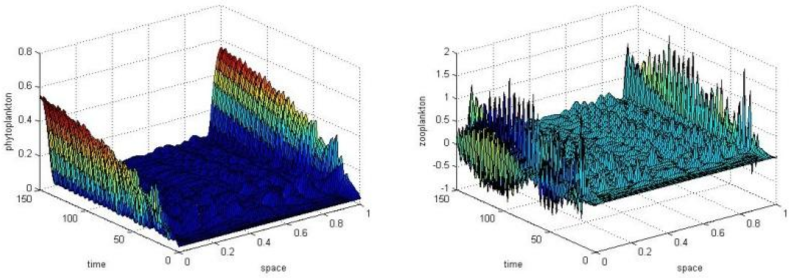


(c)

Fig. 3. Spatiotemporal patterns of phytoplankton and zooplankton density for model system (3) with (a) μ=0.09, (b) μ=0.12, (c) μ=0.35.



(a)



(b)

Fig.4. Spatiotemporal patterns of phytoplankton and zooplankton density for model system (3) with (a) w_1=5.45, (b) w_1=5.95 at μ=0.35.

VI. CONCLUSION

The role of prey taxis in presence diffusion for phytoplankton- zooplankton system is effectively studied. For this purpose, Monod-Haldane type functional response is used in our study that shows the defensive ability of prey organism. In a limited spatial domain, prey taxis play a key factor that helps to enhance the population of zooplankton. Analytically, the stability criteria of the temporal, as well as spatial models are investigated. Turing instability region has discussed in Fig. 1. Our numerical investigation reveals that the increasing value of prey taxis (i.e.μ) changes the system dynamics from stable focus to chaotic (c.f., Fig. 2). It has also observed that the coefficient of prey taxis (i.e. μ) and rate of inverse measure of the inhibitory effect of zooplankton (i.e. w_1) makes the Spatio-temporal more complex and system exhibit chaotic behavior (c.f., Fig. 3and 4). From the present study, we may conclude that the parametric value of prey taxis has a vital role in the wetland ecosystem and strongly affected the dynamics of plankton species.

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